# On local newforms for unramified U(2,1) (Automorphic forms and automorphic L-functions)



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数理解析研究所講究録
第 1826 巻 2013 年 7-17



# On local newforms for unramified $\mathrm{U}(2,1)$

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## 1 Introduction

Local newforms play an important role in the theory of automorphic representations. Roughly speaking, a newform for an irreducible generic representation $\pi$ of a $p$-adic group is a vector which attains the $L$-factor of $\pi$ via Rankin-Selberg integral. The existence of newforms was known only for $\mathrm{GL}(n)$ until the work of Roberts and Schmidt [11] for $\mathrm{GSp}(4)$. In this note, we study newform theory for unramified $\mathrm{U}(2,1)$.

This note is a survey of the author's work [6], [7], [9], [8] on newrofms for unramified $\mathrm{U}(2,1)$. Let $G$ denote the unramified unitary group in three variables defined over a $p$-adic field of odd residual characteristic. Newforms for an irreducible generic representation $(\pi, V)$ of $G$ is defined by using a family of open compact subgroups $\{K_n\}_{n\geq 0}$ of $G$, which is an analog of paramodular subgroups of $\mathrm{GSp}(4)$. For each non-negative integer $n$, we denote by $V(n)$ the space of $K_n$-fixed vectors. The smallest integer such that $V(n)$ is not trivial is called the conductor of $\pi$. We write $N_\pi$ for the conductor of $\pi$, and call $V(N_\pi)$ the space of newforms for $\pi$. An algebraic structure of $V(n)$ was studied in [6] and [9], for example, the multiplicity one theorem for newforms and the dimension formula for $V(n)$, $n\geq N_\pi$.

Our main concern is the relation of newforms and Rankin-Selberg factors. Gelbart and Piatetski-Shapiro in [4] attached a family of Rankin-Selberg integrals to an irreducible generic representation $\pi$ of $G$, and defined $L$ and $\varepsilon$-factors for $\pi$. In *loc. cit.* they showed that the spherical vector attains the $L$-factor of $\pi$ when $\pi$ is an unramified principal series representation. But there were no results for ramified representations. In this note, we establish a theory of newforms for Gelbart and Piatetski-Shapiro's integral. We see that (i) the newform for an irreducible generic representation $\pi$ of $G$ attains the $L$-factor of $\pi$ (Theorem 4.1) (ii) the conductor of $\pi$ coincides with the exponent of $q^{-s}$ of the $\varepsilon$-factor, where $q$ denotes the cardinality of the residue field (Theorem 4.3).

We summarize the contents of this paper. In section 2, we recall from [1] the theory of Rankin-Selberg integral introduced by Gelbart, Piatetski-Shapiro and Baruch. In section 3, we define newforms for $G$ and recall their basic properties. In section 4, we show Theorems 4.1 and 4.3 assuming Lemma 4.2, which is proved in section 5.

# 2 Rankin-Selberg integral

In this section, we recall from [1] the theory of Rankin-Selberg integral for $\mathrm{U}(2,1)$ introduced by Gelbart, Piatetski-Shapiro and Baruch.

## 2.1 Notation

We use the following notation. Let $F$ be a non-archimedean local field of characteristic zero, $\mathfrak{o}_F$ its ring of integers, and $\mathfrak{p}_F$ the maximal ideal in $\mathfrak{o}_F$. We fix a uniformizer $\varpi_F$ in $F$, and denote by $|\cdot|_F$ the absolute value of $F$ normalized so that $|\varpi_F| = q_F^{-1}$, where $q_F$ is the cardinality of the residue field $\mathfrak{o}_F/\mathfrak{p}_F$. Throughout this paper, we assume that the characteristic of $\mathfrak{o}_F/\mathfrak{p}_F$ is different from two.

Let $E = F[\sqrt{\epsilon}]$ be the quadratic unramified extension over $F$, where $\sqrt{\epsilon}$ is a non-square unit in $\mathfrak{o}_F$. We denote by $\mathfrak{o}_E$, $\mathfrak{p}_E$ the analogous objects for $E$. Then $\varpi_F$ is a uniformizer of $E$, and the cardinality of $\mathfrak{o}_E/\mathfrak{p}_E$ is equal to $q_F^2$. So we abbreviate $\varpi = \varpi_F$ and $q = q_F$. We realize (the group of $F$-points of) the unramified unitary group in three variables defined over $F$ as

$$G = \mathrm{U}(2,1) = \{g \in \mathrm{GL}_3(E) \mid {}^t\overline{g}Jg = J\}.$$

Here we denotes by $^-$ the non-trivial element in $\mathrm{Gal}(E/F)$ and

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $B$ be the upper triangular Borel subgroup of $G$, $U$ its unipotent radical, and $T$ the group of the diagonal elements in $G$. For a non-trivial additive character $\psi_E$ of $E$, we also denote by $\psi_E$ the character of $U$ defined by $\psi_E(u) = \psi_E(u_{12})$, for $u = (u_{ij}) \in U$. For an irreducible generic representation $(\pi, V)$ of $G$, we write $\mathcal{W}(\pi, \psi_E)$ for the Whittaker model of $\pi$ associated to $(U, \psi_E)$.

## 2.2 Zeta integrals

Let $\mathcal{C}_c^\infty(F^2)$ be the space of locally constant, compactly supported functions on $F^2$. For an irreducible generic representation $(\pi, V)$ of $G$, Gelbart and Piatetski-Shapiro introduced a family of zeta integrals which has the form $Z(s, W, \Phi)$ ($W \in \mathcal{W}(\pi, \psi_E)$, $\Phi \in \mathcal{C}_c^\infty(F^2)$) as follows:

We identify the subgroup

$$H = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in G \right\}$$

of $G$ with $\mathrm{U}(1,1)$. Since $\mathrm{SU}(1,1)$ is isomorphic to $\mathrm{SL}_2(F)$, we can write any element $h$ in $H$ as

$$h = \begin{pmatrix} b & 0 \\ 0 & \bar{b}^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} h_1 \begin{pmatrix} \sqrt{\epsilon}^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

where $b \in E^\times$ and $h_1 \in \mathrm{SL}_2(F)$. For $\Phi \in \mathcal{C}_c^\infty(F^2)$ and $h \in H$, we define a function $f(s,h,\Phi)$ on $\mathbf{C}$ by

$$f(s,h,\Phi) = |b|_E^s \int_{F^\times} \Phi((0,r)h_1)|r|_E^s d^\times r$$

by using the decomposition of $h$ in (2.1). We note that the definition of $f(s,h,\Phi)$ is independent of the choices of $b \in E^\times$ and $h_1 \in \mathrm{SL}_2(F)$.

Set $B_H = B \cap H$ and $U_H = U \cap H$. Then $U_H$ is the unipotent radical of the Borel subgroup $B_H$ of $H = \mathrm{U}(1,1)$. For $W \in \mathcal{W}(\pi, \psi_E)$ and $\Phi \in \mathcal{C}_c^\infty(F^2)$, we define zeta integral $Z(s,W,\Phi)$ by

$$Z(s,W,\Phi) = \int_{U_H \backslash H} W(h) f(s,h,\Phi) dh.$$

Then $Z(s,W,\Phi)$ absolutely converges to a function in $\mathbf{C}(q^{-2s})$ if $\mathrm{Re}(s)$ is sufficiently large.

## 2.3 $L$ and $\varepsilon$-factors

We recall the definition of $L$ and $\varepsilon$-factors attached to an irreducible generic representation $(\pi, V)$ of $G$. Set

$$I_\pi = \langle Z(s,W,\Phi) \mid W \in \mathcal{W}(\pi,\psi_E), \Phi \in \mathcal{C}_c^\infty(F^2), \psi_E : \text{non-trivial} \rangle.$$

Then $I_\pi$ is a fractional ideal of $\mathbf{C}[q^{-2s}, q^{2s}]$ which contains 1. Thus, there exists a polynomial $P(X)$ in $\mathbf{C}[X]$ such that $P(0) = 1$ and $I_\pi = (1/P(q^{-2s}))$. We define *the $L$-factor $L(s,\pi)$ of $\pi$* by

$$L(s,\pi) = \frac{1}{P(q^{-2s})}.$$

To define $\varepsilon$-factor of $\pi$, we recall the functional equation. Let $\psi_F$ be a non-trivial additive character of $F$. For $\Phi \in \mathcal{C}_c^\infty(F^2)$, we denote by $\hat{\Phi}$ its Fourier transform with respect to $\psi_F$. Then there exists $\gamma(s,\pi,\psi_F,\psi_E) \in \mathbf{C}(q^{-2s})$ which satisfies

$$\gamma(s,\pi,\psi_F,\psi_E) Z(s,W,\Phi) = Z(1-s,W,\hat{\Phi}),$$

for all $W \in \mathcal{W}(\pi, \psi_E)$ and $\Phi \in \mathcal{C}_c^\infty(F^2)$.

By using the above functional equation, we define *the* $\varepsilon$*-factor* $\varepsilon(s, \pi, \psi_F, \psi_E)$ *of* $\pi$ by

$$\varepsilon(s, \pi, \psi_F, \psi_E) = \gamma(s, \pi, \psi_F, \psi_E)\frac{L(s, \pi)}{L(1-s, \widetilde{\pi})},$$

where $\widetilde{\pi}$ is the contragradient representation of $\pi$. By [7], we obtain $L(s, \widetilde{\pi}) = L(s, \pi)$, and hence

$$\varepsilon(s, \pi, \psi_F, \psi_E) = \gamma(s, \pi, \psi_F, \psi_E)\frac{L(s, \pi)}{L(1-s, \pi)}. \tag{2.2}$$

Thus, we can show the following proposition by the standard argument:

**Proposition 2.3.** *The* $\varepsilon$*-factor* $\varepsilon(s, \pi, \psi_F, \psi_E)$ *is a monomial in* $q^{-2s}$ *of the form*

$$\varepsilon(s, \pi, \psi_F, \psi_E) = \pm q^{-2n(s-1/2)},$$

*with some* $n \in \mathbf{Z}$.

# 3 Newforms

In this section, we introduce a family of open compact subgroups of $G$, and define the notion of newforms for irreducible generic representations of $G$. We summarize the basic properties of newforms for $G$, which are an analog of those for $\mathrm{GL}(n)$ and $\mathrm{GSp}(4)$.

## 3.1 Newforms

Newforms for $G$ are defined by the following open compact subgroups $\{K_n\}_{n\geq 0}$ of $G$. For each non-negative integer $n$, we define an open compact subgroup $K_n$ of $G$ by

$$K_n = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1+\mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix} \cap G.$$

*Remark* 3.1. The definition of $K_n$ is inspired by the paramodular subgroups of $\mathrm{GSp}(4)$, which is used in [11]. We also note that the group $\begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1+\mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix}^\times$ is a conjugate of the subgroup of $\mathrm{GL}_3(E)$ which is used to define newforms for $\mathrm{GL}_3(E)$ in [5].

For an irreducible generic representation $(\pi, V)$ of $G$, we set

$$V(n) = \{v \in V \mid \pi(k)v = v,\ k \in K_n\},\ n \geq 0.$$

Then it follows from [9] that there exists a non-negative integer $n$ such that $V(n)$ is not zero.

**Definition 3.2.** We define *the conductor of* $\pi$ by

$$N_\pi = \min\{n \geq 0 \mid V(n) \neq \{0\}\}.$$

We call $V(N_\pi)$ the space of *newforms* for $\pi$ and $V(n)$ that of *oldforms*, for $n > N_\pi$.

## 3.2 Basic properties of newforms

We recall some basic properties of newforms from [6] and [9]. Firstly, the growth of dimensions of oldforms for generic representations $\pi$ is independent of $\pi$, as in the cases of $\mathrm{GL}(n)$ and $\mathrm{GSp}(4)$ (see [2], [10], [11]). The following dimension formula for oldforms holds:

**Proposition 3.3** ([6], [9]). *Let $(\pi, V)$ be an irreducible generic representation of $G$. For $n \geq N_\pi$, we have*

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1.$$

*In particular, $V(N_\pi)$ and $V(N_\pi + 1)$ are one-dimensional.*

Secondly, newforms for $G$ are test vectors for the Whittaker functional. We say that a function $W$ in $\mathcal{W}(\pi, \psi_E)$ is a newform if $W$ is fixed by $K_{N_\pi}$. The following proposition is important to the application to the theory of zeta integral:

**Proposition 3.4** ([6]). *Suppose that the conductor of $\psi_E$ is $\mathfrak{o}_E$. Then for all nonzero newforms $W$ in $\mathcal{W}(\pi, \psi_E)$, we have*

$$W(1) \neq 0.$$

## 3.3 Zeta integral of newforms

We apply newforms for $G$ to the theory of zeta integral. We suppose that the conductor of $\psi_E$ is $\mathfrak{o}_E$. One of the nice properties of the subgroups $\{K_n\}_{n\geq 0}$ is that $K_{n,H} = K_n \cap H$ is a maximal compact subgroup of $H$ for all $n$. Set $T_H = T \cap H$. Then we have an Iwasawa decomposition $H = U_H T_H K_{n,H}$, for any $n$. There exists an isomorphism

$$t : E^\times \simeq T_H; a \mapsto \begin{pmatrix} a & & \\ & 1 & \\ & & \bar{a}^{-1} \end{pmatrix}$$

For $W \in \mathcal{W}(\pi, \psi_E)$ and $\Phi \in \mathcal{C}_c^\infty(F^2)$, we obtain

$$Z(s, W, \Phi) = \int_{E^\times} \int_{K_{n,H}} W(t(a)k) f(s, k, \Phi) |a|_E^{s-1} dk d^\times a.$$

For $n \geq 0$, we denote by $\Phi_n$ the characteristic function of $\mathfrak{p}_F^n \oplus \mathfrak{o}_F$. If $W$ a newform in $\mathcal{W}(\pi, \psi_E)$, then we have

$$Z(s, W, \Phi_{N_\pi}) = \mathrm{vol}(K_{n,H}) Z(s, W) L_E(s, 1). \tag{3.5}$$

Here $L_E(s, 1) = 1/(1 - q^{-2s})$ is the $L$-factor of the trivial representation of $E^\times$ and

$$Z(s, W) = \int_{E^\times} W(t(a)) |a|_E^{s-1} d^\times a. \tag{3.6}$$

We note that Proposition 3.4 implies that the integral $Z(s, W)$ does not vanish for any non-zero newforms in $\mathcal{W}(\pi, \psi_E)$.

If $\psi_F$ has conductor $\mathfrak{o}_F$, then we have $\hat{\Phi}_{N_\pi} = q^{-N_\pi} \mathrm{ch}_{\mathfrak{o}_F \oplus \mathfrak{p}_F^{-N_\pi}}$, and hence

$$Z(1 - s, W, \hat{\Phi}_{N_\pi}) = q^{-2N_\pi(s-1/2)} Z(1 - s, W, \Phi_{N_\pi}) \tag{3.7}$$

by (3.5).

# 4 Main results

In this section, we show our two main theorems, which describe $L$ and $\varepsilon$-factors of irreducible generic representations of $G$ in terms of newforms and conductors.

## 4.1 $L$-factors and newforms

We show that zeta integrals of newforms attain $L$-factors. We normalize Haar measures on $E^\times$ and $K_{n,H}$ so that the volumes of $\mathfrak{o}_E^\times$ and of $K_{n,H}$ are one respectively. Then the following holds:

**Theorem 4.1** ([8]). *Suppose that $\psi_E$ has conductor $\mathfrak{o}_E$. Let $\pi$ be an irreducible generic representation of $G$ and $W$ the newform in $\mathcal{W}(\pi, \psi_E)$ such that $W(1) = 1$. Then we have*

$$Z(s, W, \Phi_{N_\pi}) = L(s, \pi).$$

Theorem 4.1 is reduced to the following lemma:

**Lemma 4.2.** *With the notation as above, we have*

$$Z(s, W, \Phi_{N_\pi})/L(s, \pi) = 1 \text{ or } 1/L_E(s, 1).$$

We postpone the proof of Lemma 4.2 to the next section.

*Proof of Theorem 4.1.* We further assume that $\psi_F$ has conductor $\mathfrak{o}_F$. Suppose that $Z(s, W, \Phi_{N_\pi})/L(s, \pi) = 1/L_E(s, 1)$. Then by (2.2), we obtain

$$\begin{aligned} \varepsilon(s, \pi, \psi_F, \psi_E) &= \gamma(s, \pi, \psi_F, \psi_E)\frac{L(s, \pi)}{L(1-s, \pi)} \\ &= \frac{Z(1-s, W, \hat{\Phi}_{N_\pi})}{Z(s, W, \Phi_{N_\pi})}\frac{L(s, \pi)}{L(1-s, \pi)} \\ &= q^{-2N_\pi(s-1/2)}\frac{L_E(s, 1)}{L_E(1-s, 1)}. \end{aligned}$$

The last equality follows from (3.7). This contradicts Proposition 2.3 which implies that $\varepsilon(s, \pi, \psi_F, \psi_E)$ is monomial. Thus we get $Z(s, W, \Phi_{N_\pi}) = L(s, \pi)$, as required. □

## 4.2 $\varepsilon$-factors and conductors

We show that the exponent of $q^{-2s}$ of the $\varepsilon$-factor of an irreducible generic representation $\pi$ of $G$ coincides with the conductor of $\pi$. Applying the argument in the proof of Theorem 4.1, we obtain the following:

**Theorem 4.3** ([8]). *Suppose that $\psi_E$ and $\psi_F$ have conductors $\mathfrak{o}_E$ and $\mathfrak{o}_F$ respectively. For any irreducible generic representation $\pi$ of $G$, we have*

$$\varepsilon(s, \pi, \psi_F, \psi_E) = q^{-2N_\pi(s-1/2)}.$$

# 5 Proof of Lemma 4.2

In this section, we explain how to prove Lemma 4.2.

## 5.1 Evaluation of $L$-factors

We shall evaluate $L(s,\pi)$, for each irreducible generic representation $(\pi, V)$ of $G$. The $L$-factor $L(s,\pi)$ is defined as the greatest common divisor of the zeta integrals $Z(s,W,\Phi)$. For $W \in \mathcal{W}(\pi,\psi_E)$ and $\Phi \in \mathcal{C}_c^\infty(F^2)$, there exist $W_i \in \mathcal{W}(\pi,\psi_E)$ and $\Phi_i \in \mathcal{C}_c^\infty(F^2)$ $(1 \le i \le m)$ such that

$$Z(s,W,\Phi) = \sum_{i=1}^{m} Z(s,W_i)f(s,1,\Phi_i).$$

By the theory of zeta integral for GL(1), we have

$$f(s,1,\Phi_i) \in L_E(s,1)\mathbf{C}[q^{-2s},q^{2s}].$$

Recall that we defined

$$Z(s,W) = \int_{E^\times} W(t(a))|a|_E^{s-1}d^\times a,$$

for $W \in \mathcal{W}(\pi,\psi_E)$. To estimate $Z(s,W)$, we can apply the theory of Kirillov model for GL(2).

An irreducible generic representation of $G$ is supercuspidal, or else a subrepresentation of a parabolically induced representation from $B$. The Levi component $T$ of $B$ is isomorphic to $E^\times \times \mathrm{U}(1)$. For a quasi-character $\mu_1$ of $E^\times$ and a character $\mu_2$ of U(1), we denote by $\mathrm{Ind}_B^G\mu_1 \otimes \mu_2$ the corresponding parabolically induced representation. According to the classification of representations of $G$, we have the following evaluation of the shape of $L$-factors:

**Proposition 5.1.** *Let $\pi$ be an irreducible generic representation of $G$.*

*(i) If $\pi$ is supercuspidal, then $L(s,\pi)$ divides $L_E(s,1)$.*

*(ii) If $\pi$ is a proper submodule of $\mathrm{Ind}_B^G\mu_1 \otimes \mu_2$, then $L(s,\pi)$ divides $L_E(s,\mu_1)L_E(s,1)$.*

*(iii) If $\pi = \mathrm{Ind}_B^G\mu_1 \otimes \mu_2$, then $L(s,\pi)$ divides $L_E(s,\mu_1)L_E(s,\overline{\mu_1}^{-1})L_E(s,1)$.*

## 5.2 Calculation of zeta integral of newforms

Let $W$ be the newform in $\mathcal{W}(\pi,\psi_E)$ such that $W(1) = 1$. We shall compute $Z(s,W,\Phi_{N_\pi})$. Suppose that $\pi$ has conductor zero. Then $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes 1)$, for some unramified quasi-character $\mu_1$ of $E^\times$. In this case, newforms in $\mathcal{W}(\pi,\psi_E)$ are just spherical Whittaker functions. In [4], Gelbart and Piatetski-Shapiro showed that

$$Z(s,W,\Phi_0) = L_E(s,\mu_1)L_E(s,\overline{\mu_1}^{-1})L_E(s,1)$$

by using Casselman-Shalika's formula for spherical Whittaker functions in [3]. We therefore obtain $Z(s, W, \Phi_0) = L(s, \pi)$ because of Proposition 5.1.

From now on, we assume that $N_\pi$ is positive. By (3.5), we have

$$Z(s, W, \Phi_{N_\pi}) = Z(s, W)L_E(s, 1),$$

and hence it is enough to compute $Z(s, W)$. One can easily observe that

$$Z(s, W) = \int_{E^\times} W(t(a))|a|_E^{s-1} d^\times a = \sum_{i=0}^{\infty} W(t(\varpi^i))q^{2i(1-s)}. \tag{5.2}$$

So we shall give a recursion formula for $W(t(\varpi^i))$, $i \geq 0$, in terms of two "Hecke eigenvalues" $\lambda$ and $\nu$.

We abbreviate $N = N_\pi$. Let us define the eigenvalue $\lambda$. We define a level raising operator $\theta' : V(N) \to V(N+1)$ by

$$\theta' v = \int_{K_{N+1}} \pi(k)v dk, \ v \in V(N),$$

and a level lowering operator $\delta : V(N+1) \to V(N)$ by

$$\delta w = \int_{K_N} \pi(k)w dk, \ w \in V(N+1).$$

Since $\dim V(N) = 1$, there exists $\lambda \in \mathbf{C}$ such that

$$\lambda v = \delta\theta' v,$$

for all $v \in V(N)$.

Next, we define the eigenvalue $\nu$. Put

$$\zeta = \begin{pmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{pmatrix} \in G.$$

We define the Hecke operator $T$ on $V(N+1)$ by

$$Tv = \int_{K_{N+1}\zeta K_{N+1}} \pi(k)v dk, \ v \in V(N+1).$$

Because $\dim V(N+1) = 1$, there exists $\nu$ in $\mathbf{C}$ such that

$$Tv = \nu v,$$

for all $v \in V(N+1)$.

With the notation as above, we obtain the following recursion formula for $W(t(\varpi^i))$, $i \geq 0$.

**Proposition 5.3.** *Let $(\pi, V)$ be an irreducible generic representation of $G$ whose conductor $N_\pi$ is positive. For any newform $W$ in $\mathcal{W}(\pi, \psi_E)$, we have*

$$(\nu + q^2 - \lambda)c_i + q(\nu + q^2 - q^3)c_{i+1} = q^5 c_{i+2}, \ i \geq 0,$$
$$(\nu - q^3)c_0 = q^4 c_1,$$

*where $c_i = W(t(\varpi^i))$, $i \geq 0$.*

By (5.2) and Proposition 5.3, we can describe the zeta integral of newforms in terms of $\lambda$ and $\nu$:

**Proposition 5.4.** *Let $(\pi, V)$ be an irreducible generic representation of $G$ whose conductor $N_\pi$ is positive and $W$ its newform in $\mathcal{W}(\pi, \psi_E)$ such that $W(1) = 1$. Then we have*

$$Z(s, W) = \frac{1 - q^{-2s}}{1 - \dfrac{\nu + q^2 - q^3}{q^2} q^{-2s} - \dfrac{\nu + q^2 - \lambda}{q} q^{-4s}}.$$

*In particular,*

$$Z(s, W, \Phi_{N_\pi}) = \frac{1}{1 - \dfrac{\nu + q^2 - q^3}{q^2} q^{-2s} - \dfrac{\nu + q^2 - \lambda}{q} q^{-4s}}.$$

## 5.3 Proof of Lemma 4.2

We have seen that Lemma 4.2 holds for the unramified principal series representations.

Let $\pi$ be an irreducible generic representation of $G$. We assume that $N_\pi$ is positive. Proof of Lemma 4.2 is done by comparing Propositions 5.1 and 5.4. Suppose that $\pi$ is supercuspidal or a subrepresentation of $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$, for some ramified quasi-character $\mu_1$ of $E^\times$. Then it follows from Proposition 5.1 that $L(s, \pi) = 1$ or $L_E(s, 1)$. By definition, we have $Z(s, W, \Phi_{N_\pi})/L(s, \pi) \in \mathbf{C}[q^{-2s}, q^{2s}]$. So we get

$$Z(s, W, \Phi_{N_\pi})/L(s, \pi) = 1 \text{ or } 1/L_E(s, 1),$$

by Proposition 5.4.

Suppose that $\pi$ is a subrepresentation of $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$, for unramified $\mu_1$. Then we can regard newforms for $\pi$ as functions in $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$. Due to [6], non-zero newforms $f$ in $\pi$ satisfy $f(1) \neq 0$. By using this property of newforms, we can compute the eigenvalues $\nu$ and $\lambda$ explicitly, and Lemma 4.2 follows.

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